

On sumfree subsets of hypercubes

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Abstract

We consider the possible sizes of large sumfree sets contained in the discrete hypercube $\{1, \dots, n\}^k$, and we determine upper and lower bounds for the maximal size as n becomes large. We also discuss a continuous analogue in which our lower bound remains valid and our upper bound can be strengthened, and we consider the generalization of both problems to l -fold-sumfree sets.

1 Introduction

Given an additive group Z , we refer to $A \subset Z$ as a sumfree set if $x + y \neq z$ for all $x, y, z \in A$. (Equivalently, using the notation of sumsets, A is sumfree if $(A + A) \cap A = \emptyset$.) These sets have been of interest since at least 1916, when Schur [9] proved that the positive integers could not be partitioned into finitely many such sets.

A common problem in this topic is as follows: given a particular additive group Z (or perhaps a subset Z of an additive group), how large can a sumfree subset of Z be, and further, what sort of structure do large sumfree subsets have? This problem has been considered for $Z = \mathbb{Z}_{>0}$ [2, 3], $\mathbb{Z}/p\mathbb{Z}$ [8], general finite groups (abelian [5] and non-abelian [6]), and $\{1, \dots, n\} \subset \mathbb{Z}$ for arbitrary (usually large) n [1, 10].

The last of these cases suggests a study of the “discrete hypercube” $Z = \{1, \dots, n\}^k \subset \mathbb{Z}^k$ for $k > 1$. In particular, we would like to know how proportionately large a sumfree subset of $\{1, \dots, n\}^k$ can be when

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n is large. For this purpose, we define

$$c_k := \limsup_{n \rightarrow \infty} \frac{1}{n^k} \max\{\#S : S \in \{1, \dots, n\}^k \text{ is sumfree}\}.$$

Previous work on sumfree subsets of $\{1, \dots, n\}$ has shown that $c_1 = 1/2$. (The set of odd elements, for example, is optimally large for all n .) Let S be a sumfree subset of $\{1, \dots, n\}^k$ of size αn^k , and let $k' > k$. The inverse image S' of a natural projection from $\{1, \dots, n\}^{k'}$ to $\{1, \dots, n\}^k$ is also sumfree, and has size $\alpha n^{k'}$. Using this fact, it is clear that $c_{k'} \geq c_k$ for $k' > k$, and thus $1/2 \leq c_k \leq 1$ for all k .

The largest sumfree subsets we have observed in the square $\{1, \dots, n\}^2$ take the form of thick diagonal “stripes”; generalizing this construction, we can construct large sumfree subsets in $\{1, \dots, n\}^k$ and thus prove a general lower bound for c_k .

Theorem 1.1. *Defining c_k as above,*

$$c_k \geq 1 - \frac{2}{k!} \sum_{i=0}^{\lfloor k/3 \rfloor} (-1)^i \binom{k}{i} \left(\frac{k}{3} - i\right)^k.$$

Analysis of this lower bound yields:

Corollary 1.2. *Defining c_k as above,*

$$\lim_{k \rightarrow \infty} c_k = 1.$$

We also prove a general upper bound for c_k using a combinatorial method, although it is difficult to write this bound as an explicit function of k .

Theorem 1.3. *Defining c_k as above, let α^* be the unique root in $[1/2, 1]$ of the equation*

$$\alpha = (2 - 2\alpha) \left(1 + \sum_{i=0}^k \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i \right).$$

Then

$$c_k \leq \alpha^*.$$

Our approach to the upper bound depends on the idea that if an element of a sumfree subset $S \in \{1, \dots, n\}^k$ is the sum of many pairs of elements, none of these pairs can be in S . This means that

if S contains a certain proportion of the full set, a certain number of elements cannot belong to S , which causes a contradiction if the proportion is large.

To give an idea of the distance between our lower and upper bounds, here are the approximate bounds given by these theorems for $2 \leq k \leq 6$:

$$\begin{aligned} 0.555556 &\leq c_2 \leq 0.913875 \\ 0.666667 &\leq c_3 \leq 0.942361 \\ 0.740741 &\leq c_4 \leq 0.961192 \\ 0.796639 &\leq c_5 \leq 0.973763 \\ 0.838889 &\leq c_6 \leq 0.982208 \end{aligned}$$

Our calculations for both the lower and upper bounds involve approximating numbers of lattice points in $\{1, \dots, n\}^k$ by integrating over subsets of $[0, n]^k$. This approximation is less than exact, but the error becomes trivial compared to n^k when n is large, and thus it ultimately does not affect the value of c_k . These integrals become more complicated as k grows, but they can be calculated explicitly by induction, where counting the lattice points directly becomes cumbersome in higher dimensions.

This integral method actually suggests a non-discrete version of the problem: maximizing the volume of Lebesgue-measurable sumfree subsets of the “continuous hypercube” $[0, 1]^k \subset \mathbb{R}^k$. We will see that the bounds we calculated in Theorems 1.1 and 1.3 hold in this setting, and in fact the upper bound can be improved by applying an iteration process.

We will also discuss some results that generalize our processes to l -fold-sumfree sets; that is, sets A such that $x_1 + \dots + x_l \neq z$ for all $x_1, \dots, x_l, z \in A$. The lower bound for sumfree sets extends easily to $l > 2$; the upper bound is difficult to apply when $l > 4$, but interestingly in the $l = 3$ case it gives a bound which is explicit rather than the root of an equation.

Finally, we will present some concluding remarks, suggesting two divergent paths for future investigation in the subject.

2 Introductory lemmas

In order to bound the constants under consideration, we will need the following volume formula.

Lemma 2.1. *Given $a \in [0, k]$, the volume of the region*

$$\{(x_1, \dots, x_k) \in [0, 1]^k : x_1 + \dots + x_k \leq a\} \subset \mathbb{R}^k$$

is equal to

$$\frac{1}{k!} \sum_{i=0}^{\lfloor a \rfloor} (-1)^i \binom{k}{i} (a - i)^k.$$

Proof. This is a special case of Theorem 1 in Section I.9 of [4]. \square

Remark 2.1. The proof in [4] uses probability theory, but the formula can also be obtained directly using an inclusion-exclusion argument. The latter proof is useful in that it can easily be adapted to count the number of lattice points in the region; however, we will not use this formula, so we omit the alternate proof.

We will also need the following integral formula, easily proven by induction.

Lemma 2.2.

$$\int_c^1 dx_1 \int_{c/x_1}^1 dx_2 \int_{c/x_1 x_2}^1 dx_3 \cdots \int_{c/x_1 \cdots x_{k-1}}^1 dx_k = 1 - c \sum_{i=0}^{k-1} \frac{1}{i!} \left(\ln \frac{1}{c} \right)^i$$

Remark 2.2. In a sense, the domain of integration is a multiplicative analogue of the k -simplex. Also note that the right side of the equation approaches 0 as $k \rightarrow \infty$, since the sum is a truncated Maclaurin series for e^x evaluated at $x = -\ln c$.

Proof. Let $J(k, c)$ represent the left side of the equation. The theorem is clearly true when $k = 1$, so we proceed by induction on k . Assume the statement is true for k ; then

$$\begin{aligned}
J(k+1, c) &= \int_c^1 J\left(k, \frac{c}{x_1}\right) dx_1 \\
&= \int_c^1 \left(1 - \frac{c}{x_1} \sum_{i=0}^{k-1} \frac{1}{i!} (\ln x_1 - \ln c)^i\right) dx_1 \\
&= (1-c) - c \sum_{i=0}^{k-1} \frac{1}{i!} \int_0^{-\ln c} (\ln x_1 - \ln c)^i d(\ln x_1 - \ln c) \\
&= (1-c) - c \sum_{i=0}^{k-1} \frac{1}{(i+1)!} (-\ln c)^{i+1} \\
&= 1 - c \sum_{i=0}^{k-1} \frac{1}{i!} \left(\ln \frac{1}{c}\right)^i
\end{aligned}$$

and thus the lemma holds for all k . \square

Finally, we quote a theorem from Lang, adapted for our purposes, which will allow us to use integrals to approximate subsets of lattices.

Theorem 2.3 (Lang). *Let D be a subset of $[0, 1]^k$ such that the boundary of D has a Lipschitz-continuous parametrization in $(k-1)$ variables, and let $nD = \{nx : x \in D\}$. Then*

$$\# \left(\{1, \dots, n\}^k \cap nD \right) = n^k \text{Vol}(D) + O(n^{k-1}).$$

Proof. Apply Theorem 2, p. 128 in [7] with $L = \mathbb{Z}^k$ and $F = (0, 1]^k$. There are fewer than $k(n+1)^{k-1}$ lattice points in the intersection of nD and $(\{0, \dots, n\}^k \setminus \{1, \dots, n\}^k)$, and these can be absorbed into the error term. \square

3 Bounding c_k from below

One method of generating sumfree sets in $\{1, \dots, n\}$ is to consider “cross-section” sets

$$K_a := \{(x_1, \dots, x_k) \in \{1, \dots, n\}^k : x_1 + \dots + x_k = a\}.$$

If A is a sumfree set in $\{k, \dots, kn\}$, the set $S = \cup_{a \in A} K_a$ is sumfree, because if (x_1, \dots, x_k) and (y_1, \dots, y_k) are both contained in S , then

$$(x_1 + y_1) + \dots + (x_k + y_k) = (x_1 + \dots + x_k) + (y_1 + \dots + y_k) \notin A,$$

so the sum of these two elements is not in S .

We will determine a lower bound for c_k using sets of the form

$$S(n, k, a) := \{(x_1, \dots, x_k) \in \{1, \dots, n\}^k : a \leq x_1 + \dots + x_k < 2a\}.$$

Since $\{a+1, \dots, 2a\}$ is clearly sumfree in $\{k, \dots, kn\}$, $S(n, k, a)$ is sumfree. To obtain an optimal lower bound for this method, we need to choose a value of a that maximizes the size of $S(n, k, a)$. We approximate this size using the region

$$\tilde{S}(n, k, a) := \{(x_1, \dots, x_k) \in [0, n]^k \subset \mathbb{R}^k : a \leq x_1 + \dots + x_k < 2a\}.$$

Note that since $\tilde{S}(1, k, a)$ is just a scaled-down copy of $\tilde{S}(n, k, an)$, we have

$$\tilde{S}(n, k, an) = n^k \tilde{S}(1, k, a).$$

Proof of Theorem 1.1. By Lemma 2.1, the volume of

$$\{(x_1, \dots, x_k) \in [0, 1]^k \subset \mathbb{R}^k : x_1 + \dots + x_k < a\}$$

is equal to

$$V_1(k, a) := \frac{1}{k!} \sum_{i=0}^{\lfloor a \rfloor} (-1)^i \binom{k}{i} (a-i)^k.$$

Changing variables, the volume of

$$\{(x_1, \dots, x_k) \in [0, 1]^k \subset \mathbb{R}^k : x_1 + \dots + x_k > 2a\}$$

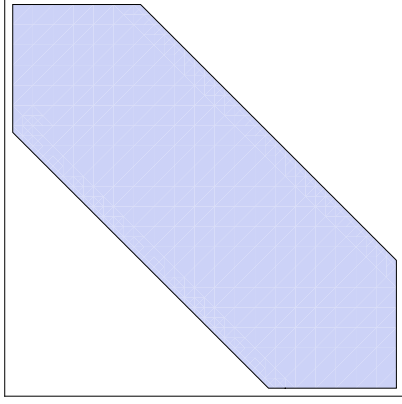
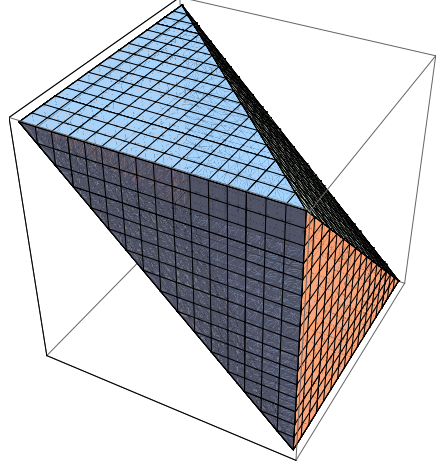
is equal to

$$V_2(k, a) := \frac{1}{k!} \sum_{i=0}^{\lfloor k-2a \rfloor} (-1)^i \binom{k}{i} (k-2a-i)^k.$$

We wish to choose a value of a (for each k) which maximizes

$$\text{Vol}(\tilde{S}(1, k, a)) = 1 - V_1(k, a) - V_2(k, a).$$

A computer search (for $k < 60$) suggests the optimal choice satisfies $a = k/3 + O(1)$, although it is difficult to determine an exact formula. For our lower bound, we choose $a = k/3$; this value appears to be close to optimal, and it gives a concise expression for $\text{Vol}(\tilde{S}(1, k, a))$ (since $V_1(k, k/3) = V_2(k, k/3)$). The regions $\tilde{S}(1, k, k/3)$ for $k = 2, 3$ are shown in Figures 1 and 2.

Figure 1: Sumfree region for $k = 2$ Figure 2: Sumfree region for $k = 3$

By Theorem 2.3, $\#S(n, k, kn/3) = \text{Vol}(\tilde{S}(n, k, kn/3)) + O(n^{k-1})$, since all of the boundaries of the region are hyperplanes and are thus Lipschitz parametrizable. Then we have

$$\begin{aligned}
 c_k &\geq \limsup_{n \rightarrow \infty} \frac{1}{n^k} \#(S(n, k, kn/3)) \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{n^k} \left(\text{Vol}(\tilde{S}(n, k, kn/3)) + O(n^{k-1}) \right) \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{n^k} \left(n^k \text{Vol}(\tilde{S}(1, k, k/3)) + O(n^{k-1}) \right) \\
 &= \text{Vol}(\tilde{S}(1, k, k/3)) \\
 &= 1 - \frac{2}{k!} \sum_{i=0}^{\lfloor k/3 \rfloor} (-1)^i \binom{k}{i} \left(\frac{k}{3} - i \right)^k.
 \end{aligned}$$

□

To determine the behavior of this lower bound, we need the following lemma.

Lemma 3.1. *Let a, b satisfy $0 < a < b < \frac{1}{2}$ and*

$$\frac{1}{3} - a < \frac{b^b(1-b)^{(1-b)}}{e}.$$

Then

$$\lim_{k \rightarrow \infty} \frac{1}{k!} \sum_{i=\lceil ak \rceil}^{\lfloor bk \rfloor} \binom{k}{i} \left(\frac{k}{3} - i \right)^k = 0.$$

Proof.

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k!} \sum_{i=\lceil ak \rceil}^{\lfloor bk \rfloor} \binom{k}{i} \left(\frac{k}{3} - i \right)^k &\leq \lim_{k \rightarrow \infty} \frac{1}{k!} \sum_{i=\lceil ak \rceil}^{\lfloor bk \rfloor} \binom{k}{bk} \left(\frac{k}{3} - ak \right)^k \\ &\leq \lim_{k \rightarrow \infty} \frac{(b-a+1)k}{(bk)!(k-bk)!} k^k \left(\frac{1}{3} - a \right)^k. \end{aligned}$$

Then, using Stirling's approximation,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{(b-a+1)k}{(bk)!(k-bk)!} k^k \left(\frac{1}{3} - a \right)^k &= \lim_{k \rightarrow \infty} \frac{b-a+1}{2\pi\sqrt{b(1-b)}} \left(\frac{(\frac{1}{3}-a)e}{b^b(1-b)^{(1-b)}} \right)^k \\ &= 0. \end{aligned}$$

□

Proof of Corollary 1.2. Define a sequence $\{a_i\}$ as follows:

$$\begin{aligned} a_0 &= \frac{1}{3}, \\ a_{i+1} &= \frac{1}{3} - \frac{a_i^{a_i}(1-a_i)^{(1-a_i)}}{e}. \end{aligned}$$

Calculating the initial terms of this sequence, we find that $a_7 < 0$, and that a_1, a_2, \dots, a_6 are irrational. Thus we can split the sum as follows:

$$\sum_{i=0}^{\lfloor k/3 \rfloor} (-1)^i \binom{k}{i} \left(\frac{k}{3} - i \right)^k = \sum_{i=\lceil a_1 k \rceil}^{\lfloor a_0 k \rfloor} + \sum_{i=\lceil a_2 k \rceil}^{\lfloor a_1 k \rfloor} + \dots + \sum_{i=0}^{\lfloor a_6 k \rfloor}.$$

By Lemma 3.1, each of these partial sums approaches zero as k approaches infinity, and so the entire sum does as well. This means that the lower bound determined in Theorem 1.1 approaches 1 as k grows, and therefore so does c_k . □

4 Bounding c_k from above

The process of finding an upper bound for c_k is a bit more complicated, since we cannot do so simply by exhibiting a sumfree set. Here our procedure is to assume our sumfree set has a certain size, and from this we determine a contradiction if the set is too large.

Proof of Theorem 1.3. Let S be a sumfree subset of $\{1, \dots, n\}^k$ with $\#S \geq \alpha n^k$. Suppose $b = (b_1, \dots, b_k)$ is an element of S with component values close to 1. There are $\frac{1}{2} \prod_{i=1}^k (b_i - 1)$ disjoint pairs of elements in $\{1, \dots, n\}^k$ which sum to b , unless all of the b_i 's are even, in which case there are $\frac{1}{2}(\prod_{i=1}^k (b_i - 1) + 1)$, to account for the point $(\frac{b_1}{2}, \dots, \frac{b_k}{2})$. Either way, the number of pairs is equal to $\frac{1}{2}b_1 \cdots b_k + O(n^{k-1})$.

At least one element from each of these pairs must be absent from S , so (approximately)

$$\alpha n^k \leq \#S \leq n^k - \frac{1}{2}b_1 b_2 \cdots b_k + O(n^{k-1})$$

and thus

$$b_1 b_2 \cdots b_k \leq (2 - 2\alpha)n^k + O(n^{k-1}) = \beta n^k,$$

where $\beta = (2 - 2\alpha) + O(1/n)$.

This “disqualifies” a number of lattice points from being contained in S , namely

$$T(n, k, \alpha) = \{(b_1, \dots, b_k) \in \{1, \dots, n\}^k : b_1 b_2 \cdots b_k > \beta n^k\}.$$

As in the last section, we will approximate this collection of lattice points by the region

$$\tilde{T}(n, k, \alpha) = \{(b_1, \dots, b_k) \in [0, n]^k \subset \mathbb{R}^k : b_1 b_2 \cdots b_k > \beta n^k\}.$$

We can calculate the volume of $\tilde{T}(n, k, \alpha)$ using an integral:

$$\begin{aligned} \text{Vol}(\tilde{T}(n, k, \alpha)) &= \int_{\beta n}^n dx_1 \int_{\beta n/x_1}^n dx_2 \int_{\beta n/x_1 x_2}^n dx_3 \cdots \int_{\beta n/x_1 \cdots x_{k-1}}^n dx_k \\ &= n^k \int_{\beta}^d x_1 \int_{\beta/x_1}^d x_2 \int_{\beta/x_1 x_2}^d x_3 \cdots \int_{\beta/x_1 \cdots x_{k-1}}^d x_k \\ &= n^k \left(1 - \beta \sum_{i=0}^{k-1} \frac{1}{i!} \left(\ln \frac{1}{\beta} \right)^i \right), \end{aligned}$$

using Lemma 2.2 with $c = \beta$ in the final step. Since, by Theorem 2.3, $\text{Vol}(T(n, k, \alpha)) = \text{Vol}(\tilde{T}(n, k, \alpha)) + O(n^{k-1})$, this indicates that for any α such that

$$\alpha = 1 - \frac{1}{2}\beta + O(1/n) \geq f(\beta) := \beta \sum_{i=0}^{k-1} \frac{1}{i!} \left(\ln \frac{1}{\beta} \right)^i,$$

any set larger than αn^k is simultaneously smaller than αn^k , yielding a contradiction.

Observe that

$$\begin{aligned} f'(\beta) &= \beta \left(\sum_{i=0}^{k-2} \frac{1}{i!} \left(\ln \frac{1}{\beta} \right)^i \right) \left(-\frac{1}{\beta} \right) + \sum_{i=0}^{k-1} \frac{1}{i!} \left(\ln \frac{1}{\beta} \right)^i \\ &= \frac{1}{(k-1)!} \left(\ln \frac{1}{\beta} \right)^{k-1} > 0. \end{aligned}$$

Thus, as β increases from 0 to 1, $f(\beta)$ increases monotonically from 0 to 1, while $(1 - \beta/2)$ decreases monotonically from 1 to 1/2. Therefore, the equation $1 - \beta/2 = f(\beta)$ has a unique root $\beta^* \in [0, 1]$, and letting $\alpha^* = (1 - \beta^*/2)$, we must have $\alpha < \alpha^* + O(1/n)$ to avoid a contradiction. Letting n approach infinity, we conclude that $c_k \leq \alpha^*$. \square

5 A continuous analogue

In the previous two sections, we used the volume of continuous regions to estimate the size of discrete sets. Alternatively, we could have asked our question about the continuous regions in the first place. Let us consider

$$\tilde{c}_k := \max\{\text{Vol}(S) : S \in [0, 1]^k \text{ is measurable and sumfree}\}.$$

Theorem 5.1. *Defining \tilde{c}_k as above,*

$$\tilde{c}_k \geq 1 - \frac{2}{k!} \sum_{i=0}^{\lfloor k/3 \rfloor} (-1)^i \binom{k}{i} \left(\frac{k}{3} - i \right)^k.$$

Corollary 5.2. *Defining \tilde{c}_k as above,*

$$\lim_{k \rightarrow \infty} \tilde{c}_k = 1.$$

Theorem 5.3. *Defining \tilde{c}_k as above, let α^* be the unique root in $[1/2, 1]$ of the equation*

$$\alpha = (2 - 2\alpha) \left(1 + \sum_{i=0}^k \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i \right).$$

Then

$$\tilde{c}_k \leq \alpha^*.$$

Proof. The proofs of these statements are virtually identical to the proofs of Theorem 1.1, Corollary 1.2, and Theorem 1.3 respectively. The only difference is that $S(n, k, a) = \tilde{S}(n, k, a)$ and $T(n, k, a) = \tilde{T}(n, k, a)$, so there are no error terms to incorporate.

The proof of Theorem 5.3 warrants one additional comment. If S is a Lebesgue-measurable sumfree set, and $(b_1, \dots, b_k) \in S$, then the sets

$$\begin{aligned} S_{b_1, \dots, b_k} &:= S \cap ([0, b_1] \times \dots \times [0, b_k]) \\ S'_{b_1, \dots, b_k} &:= \{(b_1, \dots, b_k) - x : x \in S_{b_1, \dots, b_k}\} \end{aligned}$$

are disjoint sets of equal volume contained in $[0, b_1] \times \dots \times [0, b_k]$. Therefore we have

$$\text{Vol}(S_{b_1, \dots, b_k}) \leq \frac{1}{2} b_1 \dots b_k.$$

This substitutes for the combinatorial argument that begins the proof of Theorem 1.3. \square

The upper bound for \tilde{c}_k may be improved by a slightly different approach. Recall the definition of f from the proof of Theorem 1.3, and suppose $\text{Vol}(S) = \alpha$, where $\alpha = f(\alpha)$. This would require S to consist of all of $[0, 1]^k$ except the “integral wedge” $\tilde{T}(n, k, a)$ that we removed from the upper right corner. But this would mean that S contains *all* of a smaller set $[0, m]^k$. Scaling by a factor of $1/k$, this violates the upper bound we’ve just determined. We can improve our upper bound by exploiting this condition and iterating the process.

Theorem 5.4. *Defining \tilde{c}_k as above, let α^{**} be the unique root in $(1/2, 1)$ of the equation*

$$\alpha = \frac{1}{2} - \alpha + \sum_{i=k}^{\infty} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i.$$

Then

$$\tilde{c}_k \leq \alpha^{**}.$$

Proof. If $\text{Vol}(S) = \alpha$, consider the set $S' = [0, (2 - 2\alpha)^{1/k}]^k \in [0, n]^k$, which is disjoint from $\tilde{T}(n, k, \alpha)$ except for a single point. Since S cannot intersect $\tilde{T}(n, k, \alpha)$, the *smallest* density $(S \cap S')/S'$ we can achieve is

$$\begin{aligned} \varphi_k(\alpha) &:= \frac{\alpha - (1 - \text{Vol}(\tilde{T}(n, k, \alpha)) - (2 - 2\alpha))}{(2 - 2\alpha)} \\ &= \frac{\text{Vol}(\tilde{T}(n, k, \alpha))}{(2 - 2\alpha)} + \frac{1}{2} \\ &= \frac{\alpha}{2 - 2\alpha} - \sum_{i=1}^{k-1} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i, \end{aligned}$$

where again we apply Lemma 2.2 in the final step.

If any $\varphi_k^m(\alpha) > 1$ (that is, the m th iteration of φ_k , not the m th power), we have a contradiction. We wish to show that the function

$$\psi_k(\alpha) = \varphi_k(\alpha) - \alpha$$

has a unique root α^{**} on the interval $(0.5, 1)$, and that any $\alpha > \alpha^{**}$ will grow larger than 1 through repeated application of ψ_k . First we observe that $\psi_k(0.5) = 0$. On the interval $(1/2, 1)$,

$$\begin{aligned} \psi_k(\alpha) &= \frac{\alpha}{2 - 2\alpha} - \left(\frac{1}{2 - 2\alpha} - 1 - \sum_{i=k}^{\infty} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i \right) - \alpha \\ &= \frac{1}{2} - \alpha + \sum_{i=k}^{\infty} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i. \end{aligned}$$

Next we determine the first and second derivatives:

$$\begin{aligned} \psi'_k(\alpha) &= \frac{1}{1 - \alpha} \sum_{i=k-1}^{\infty} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i - 1. \\ \psi''_k(\alpha) &= \frac{1}{1 - \alpha} \left(\sum_{i=k-2}^{\infty} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i \right) \\ &\quad + \frac{1}{(1 - \alpha)^2} \left(\sum_{i=k-1}^{\infty} \frac{1}{i!} \left(\ln \frac{1}{2 - 2\alpha} \right)^i \right). \end{aligned}$$

Inspecting these derivatives, we see that $\psi'_k(1/2) = -1 < 0$, and ψ''_k is positive on the interval $(1/2, 1)$. Thus, ψ_k has at most one root on

the interval. Finally, since the quantity $(\ln \frac{1}{2-2\alpha})$ approaches infinity as α approaches 1 from below, it is clear that

$$\lim_{\alpha \rightarrow 1^-} \psi'_k(\alpha) = -\infty.$$

This implies that $\psi_k(\alpha) = \varphi_k(\alpha) - \alpha$ has a root $\alpha^{**} \in (1/2, 1)$, and furthermore, since ψ_k is increasing for $\alpha > \alpha^{**}$, iteration of φ_k on any $\alpha > \alpha^{**}$ will eventually give a result larger than 1. Thus we must have $\tilde{c}_k \leq \alpha^{**}$. \square

Figure 3 illustrates the method applied to prove Theorem 5.3, in which one region of the hypercube is ruled out, while Figure 4 illustrates the method of Theorem 5.4, in which successive regions are removed from hypercubes of decreasing size.

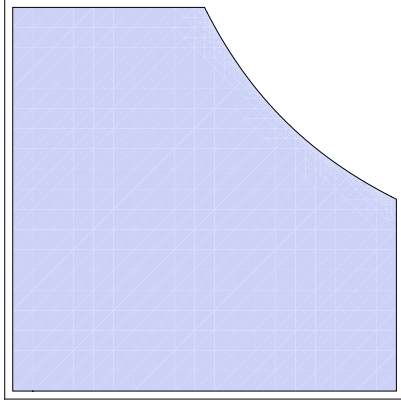


Figure 3: Method of Theorem 5.3

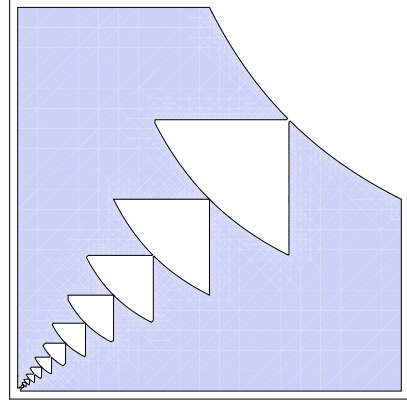


Figure 4: Method of Theorem 5.4

Theorem 5.4 yields the following bounds for $2 \leq k \leq 6$ (showing an improvement in the upper bound compared to the data presented in Section 1):

$$\begin{aligned} 0.555556 &\leq \tilde{c}_2 \leq 0.727309 \\ 0.666667 &\leq \tilde{c}_3 \leq 0.840690 \\ 0.740741 &\leq \tilde{c}_4 \leq 0.899940 \\ 0.796639 &\leq \tilde{c}_5 \leq 0.935089 \\ 0.838889 &\leq \tilde{c}_6 \leq 0.957139 \end{aligned}$$

It seems possible that this technique may also be used to improve the upper bound in the discrete case. However, the process of iteration creates serious obstacles in the translation of Theorem 5.4; every iteration introduces its own error term, and since the number of iterations is unbounded, the continuous proof is not sufficient in the discrete setting.

It is worth noting that while the constants c_k and \tilde{c}_k seem similar in nature (and indeed we apply similar methods when bounding them), there is no obvious relation between them; it is not even clear which of these values is larger for a given k .

6 Generalization to l -fold-sumfree sets

A sumfree set S is, by definition, a set such that $f(x, y, z) := x + y - z \neq 0$ for all $x, y, z \in S$. We can generalize this definition by replacing f with any other linear form $f(x_1, \dots, x_n)$ and considering sets such that this form is nonzero for any $x_1, \dots, x_n \in S$.

As a natural generalization, we call S an l -fold-sumfree set if

$$\forall x_1, \dots, x_l, z \in S, x_1 + x_2 + \dots + x_l - z \neq 0,$$

or equivalently, using sumset notation, if

$$lA \cap A = \emptyset.$$

We define

$$c_{k,l} := \limsup_{n \rightarrow \infty} \frac{1}{n^k} \max\{\#S : S \in \{1, \dots, n\}^k \text{ is } l\text{-fold sumfree}\},$$

or in the continuous setting,

$$\tilde{c}_{k,l} := \max\{\text{Vol}(S) : S \in [0, 1]^k \text{ is measurable and } l\text{-fold-sumfree}\}.$$

Remark 6.1. In some of the literature ([2], for instance), these sets are simply referred to as l -sumfree. However, this description is used with various meanings (see [10]), so we will use the term l -fold-sumfree for added clarity.

As in the sumfree ($l = 2$) case, we can construct large sumfree sets using “diagonal stripes”, leading to a similar lower bound.

Theorem 6.1. *Defining $c_{k,l}$ and $\tilde{c}_{k,l}$ as above,*

$$c_{k,l} \geq 1 - \frac{2}{k!} \sum_{i=0}^{\lfloor k/(l+1) \rfloor} (-1)^i \binom{k}{i} \left(\frac{k}{l+1} - i \right)^k,$$

$$\tilde{c}_{k,l} \geq 1 - \frac{2}{k!} \sum_{i=0}^{\lfloor k/(l+1) \rfloor} (-1)^i \binom{k}{i} \left(\frac{k}{l+1} - i \right)^k.$$

Proof. We follow the proof of Theorem 1.1, except now we use the l -fold-sumfree set

$$\tilde{S}(n, k, a) := \left\{ (x_1, \dots, x_k) \in [0, n]^k : \frac{1}{l+1} \leq x_1 + \dots + x_k < \frac{l}{l+1} \right\}.$$

The given bound is the volume of this set by Lemma 2.1. \square

Corollary 6.2. *Defining $c_{k,l}$ and $\tilde{c}_{k,l}$ as above,*

$$\lim_{k \rightarrow \infty} c_{k,l} = \lim_{k \rightarrow \infty} \tilde{c}_{k,l} = 1.$$

Proof. The lower bound for $c_{k,l}$ and $\tilde{c}_{k,l}$ given in Theorem 6.1 is larger than the lower bound for c_k given in Theorem 1.1 (as it is the volume of a larger region). Since the previous bound approaches 1 as k grows large, this one does as well. \square

Our upper bound does not extend as easily. Adapting our methods, we can deal with the $l = 3$ case, and in fact find an upper bound which is both explicit and reasonably effective; however, it is not evident how to deal with any of the cases where $l \geq 4$.

Theorem 6.3. *Defining $c_{k,l}$ and $\tilde{c}_{k,l}$ as above,*

$$c_{k,3} \leq 1 - \frac{1}{(1 + 2^{1/k})^k},$$

$$\tilde{c}_{k,3} \leq 1 - \frac{1}{(1 + 2^{1/k})^k}.$$

Proof. Let S be an l -fold-sumfree subset of $[0, 1]^k$ with $\text{Vol}(S) = \alpha$. We define the sets

$$A_1 = S \cap [0, \gamma]^k,$$

$$A_2 = S \cap [1 - \gamma, 1]^k,$$

where $(1 - \alpha)^{1/k} < \gamma < \frac{1}{2}$.

Consider the element

$$a := (1 - \gamma, 1 - \gamma, \dots, 1 - \gamma)$$

and suppose that the sets A_2 and the translation $A_1 + a \subset [1 - \gamma, 1]^k$ are disjoint. Then

$$\begin{aligned} \alpha &\leq (1 - 2\gamma) + \text{Vol}(A_1) + \text{Vol}(A_2) \\ &= (1 - 2\gamma) + \text{Vol}(A_1 + a) + \text{Vol}(A_2) \\ &\leq (1 - 2\gamma) + \gamma = 1 - \gamma, \end{aligned}$$

which contradicts our assumption on γ .

Thus, there exist elements $w, z \in S$ such that $w + a = z$. This means S cannot contain any pair of elements $x, y \in S$ such that $x + y = a$, or else we would have $w + x + y = z$, a contradiction since S is l -fold-sumfree. Then, as in the proof of Theorem 5.3, we must have

$$\text{Vol}(S \cap [1 - \gamma]^k) \leq \frac{1}{2}(1 - \gamma)^k.$$

Using this result and the restriction on γ ,

$$1 - \gamma^k < \alpha \leq 1 - \frac{1}{2}(1 - \gamma)^k,$$

and thus

$$\begin{aligned} \frac{1}{2}(1 - \gamma)^k &< \gamma^k \\ (1 - \gamma) &< \gamma \cdot 2^{1/k} \\ \frac{1}{2^{1/k}} &< \gamma. \end{aligned}$$

Finally, we use the bound on γ to bound α (and thus $\tilde{c}_{k,3}$):

$$\alpha \leq 1 - \frac{1}{2}(1 - \gamma)^k < 1 - \frac{1}{(1 + 2^{1/k})^k}.$$

We achieve the upper bound for $c_{k,3}$ using the same sort of integral approximation technique we applied to Theorems 1.1 and 1.3. The process is virtually identical, so we omit the details here. \square

Theorems 6.1 and 6.3 give us lower and upper bounds for $\tilde{c}_{k,3}$; looking at the cases where $2 \leq k \leq 6$, we get the following results:

$$\begin{aligned} 0.750000 &\leq \tilde{c}_{2,3} \leq 0.828427 \\ 0.859375 &\leq \tilde{c}_{3,3} \leq 0.913360 \\ 0.916667 &\leq \tilde{c}_{4,3} \leq 0.956464 \\ 0.949219 &\leq \tilde{c}_{5,3} \leq 0.978167 \\ 0.968620 &\leq \tilde{c}_{6,3} \leq 0.989061 \end{aligned}$$

These bounds illustrate that for 3-fold-sumfree sets, the largest “diagonal stripe” sets have very close to maximal size.

7 Concluding remarks

All of the large sumfree (and l -fold-sumfree) sets we have constructed are unions of sets of the form K_a as defined in Section 3. These are certainly the simplest sets to grasp, but there is no guarantee that the largest sumfree sets have this structure.

If we limit ourselves to these K_a -unions, the problem is simplified to choosing an optimal sumfree set $A \subset \{k, \dots, kn\}$. (To conserve space in this section, we will use the discrete notation to discuss both the continuous and discrete problems.) Since the sets K_a are not of equal size, this is a different task than finding a large sumfree set A . This suggests a more general combinatorial problem.

Question 7.1. *Given an additive set with a weight assigned to each element, what methods can we use to construct sumfree (resp. l -fold-sumfree) sets that maximize the sum of the weights of the elements?*

On the other hand, if we relax this structural constraint, we know virtually nothing about whether the upper bound on the size increases.

Question 7.2. *Are there optimally large sumfree (resp. l -fold-sumfree) subsets of $\{1, \dots, n\}^k$ which are not the union of “cross-section” sets?*

Addressing both of these questions would solve the problems we have been studying. Question 7.1 is unlikely to have an answer in full generality, although if the weight distribution is highly structured, as it is in this context, there may be methods of approach.

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